HERODOTOS AND THE ABACUS

It is perhaps natural to assume that Herodotos, like his contemporaries,[1] used an abacus for calculations.[2] When his calculations are correct,[3] however, it is impossible to trace the steps by which they were made and so to determine the means used in the calculation. But where he gives us both an incorrect solution and the material for the correct solution, it is possible not only to check his results but also to show the means he used to arrive at them.

Of six such calculations where Herodotos provides an apparently incorrect solution, only three show arithmetical error. The other three are: I, 32, where it is not Herodotos' arithmetic that is at fault, but his understanding of the calendar;[4] V, 52, where a part of the material which he gave has dropped out of the text and so leaves

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[1] Herodotos himself (II, 36, 4) speaks of the Greek method of counting with pebbles in what must be vertical columns and so presumably a form of abacus: *γράμματα γράφουσι καὶ λογίζονται ψήφοισι Ἕλληνες μὲν ἀπὸ τῶν ἀριστερῶν ἐπὶ τὰ δεξιὰ φέροντες τὴν χεῖρα.* Aischylos' phrase *ἐν ψήφῳ λέγειν* (*Agamemnon*, line 570) seems also to refer to an abacus. Two calculating references in Aristophanes, *Vespae*, suggest the physical form and use of the abacus:

lines 332-3 *ἢ δῆτα λίθον με ποίησον ἐφ' οὗ / τὰς χοιρίνας ἀριθμοῦσιν.*
lines 656-7 *καὶ πρῶτον μὲν λόγισαι φαύλως, μὴ ψήφοις, ἀλλ' ἀπὸ χειρός, / τὸν φόρον.*

The stone slab used to count the votes in the courts is mentioned in Aristotle's *Constitution of the Athenians*, 69. The use of pebble-counting for calculations too difficult to be done in the head (on the fingers) is also suggested by the fourth-century comedy-writer Alexis (Athenaios, 117 c-e) and by Demosthenes (XVIII, 229). The abacus is mentioned by Lysias (fr. 50 Thalheim): *μετ' ἀβακίου δὲ καὶ τραπεζίου πωλῶν ἑαυτόν.* As early as the sixth century Solon is supposed to have compared tyrants' men to the pebbles of an abacus:

> Diogenes Laertios, I, 59 *ἔλεγε δὲ τοὺς παρὰ τοῖς τυράννοις δυναμένους παραπλησίους εἶναι ταῖς ψήφοις ταῖς ἐπὶ τῶν λογισμῶν. καὶ γὰρ ἐκείνων ἑκάστην ποτὲ μὲν πλείω σημαίνειν, ποτὲ δὲ ἥττω· καὶ τούτων τοὺς τυράννους ποτὲ μὲν ἕκαστον μέγαν ἄγειν καὶ λαμπρόν, ποτὲ δὲ ἄτιμον.*

Much the same analogy occurs in Polybios, V, 26.

[2] The beginnings of "alphabetic" arithmetic (see, for example, T. L. Heath, *Manual of Greek Mathematics* [Oxford, 1931], pp. 28-32) can not be determined, since the date of the introduction of alphabetic numerals is still in dispute. Earlier authorities (see, for example, Nagl in P. W., *R.E.*, Suppl. III, cols. 11-12) dated their first use to the fourth century B.C. or later, but the most recent work on this subject (M. N. Tod, *B.S.A.*, XLV, 1950, p. 137), reminds us that whatever the significance and purpose of *I.G.*, I², 760, may be, it undoubtedly does use alphabetic symbols as numerals and that it must be dated by its letter-shapes to the fifth century B.C. But the continued, almost universal, use of the acrophonic numerals through at least the fourth century B.C., as well as the continuing references to the use of the abacus, makes it unlikely that alphabetic numerals were sufficiently well-established in the fifth century to have allowed the development of "alphabetic" arithmetic.

[3] Simple addition and multiplication seem to have caused him no difficulty. See the correct calculations in VII, 89, 184; VIII, 2; IX, 28.

[4] Once it is assumed that months are uniformly 30 days long and that an intercalary month was added every other year, it is arithmetically correct to conclude that the number of days in 70 years is 360 x 70 plus 30 x 35 or 25,200 plus 1,050, which equals 26,250.

us short of his totals;[5] and VIII, 48, where the probability is that one number has dropped from our text.[6] In the other three passages, our material is almost certainly complete, the text appears to be sound, and the errors can be shown to be arithmetical. It is to these that we must look for enlightenment on Herodotos' arithmetical means and methods.

II, 142: the problem is how many years are represented by the 341 kings of Egypt. Herodotos calculates as follows: " 300 generations of men make 10,000 years, for three generations of men are 100 years. Of the other 41 generations, which are over and above the 300, the number of years is 1340. Thus in 11,340 years they said there was no god in human shape." The material is given for the correct solution (11,366⅔ years) both in formula and by example: three generations equal 100 years, so 300 generations will be 100 x 100 years or 10,000 years (i. e., 300 divided by 3 equals 100; 100 x 100 is 10,000). It is to be noted that Herodotos does not treat 341 as one number but breaks it down into two parts before dividing and multiplying, so that the final result is obtained by adding the two results. It is in the second part, the 41 generations, that the mistake occurs. But if Herodotos had continued to use the same method here as for the 300 generations, it is difficult to see where his particular error could enter: 41 ÷ 3 is 13⅔; 13⅔ x 100 is 1366⅔. Motivated perhaps by the desire to avoid the fraction two-thirds, which adds still another stage to the calculation, Herodotos apparently thought it easier to multiply 41 by 33⅓. As we do this by simple arithmetic the solution is still 1366⅔. How did Herodotos do it?

$$\begin{array}{rcr} 30 \times 41 & \text{is} & 1230 \\ 3 \times 41 & \text{is} & \underline{123} \\ & & 1353 \end{array}$$

This is correct; then ⅓ x 41 is 13 plus (the fraction here may be disregarded as being less than a year). And it is here, and only here, that Herodotos could have made the mistake which gave him 1340 as a answer. He subtracted the 13 from 1353 instead of adding it, obtaining 1340 instead of 1366. This mistake shows his lack of complete sureness in dealing with fractions and does more than anything to explain why he wished to avoid the fraction two-thirds in the other method. The reason for his confusion between addition and subtraction in this case is less certain. It seems likely

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[5] Not only the fact that Herodotos' itemized *stathmoi* and parasangs do not add up to his totals, but also the absence of the parasang figure for Matiene and geographical accuracy demand an insertion like that proposed by de la Barre and transposed by Stein. Cf. W. W. How and J. Wells, *A Commentary on Herodotus*, II (Oxford, 1936), p. 23.

[6] See the note on this passage in How and Wells, *op. cit.* For our purposes it is immaterial whether Herodotos wrote ἄλλαι δύο καὶ δέκα νέες (van Herwerden) or ἄλλαι ί νέες (Cobet). The insertion is made not so much to save Herodotos' arithmetic as to bring the Aiginetan squadron into second place as far as numerical strength is concerned or to identify an Aiginetan squadron of ten with that mentioned by Aischylos.

that, having arrived at the 1353 and the 13, he was unconsciously seduced by the rightness of the round number and so subtracted to get 1340. There may also have been in his mind, if he had tried finding the multiple of three nearest to 41 for the other method, the strong feeling that from his first product he must subtract one-third of the multiplicand, i. e., 42÷3 is 14; 14 x 100 is 1400; 1400 — 33⅓ is 1366⅔. When he changed methods, that feeling, out of context though it was, may have continued to influence him into subtracting rather than adding.

That this calculation and mistake were made on the abacus still requires proof. The fact that this was the type of mistake which I frequently made when first experimenting with the abacus is no proof, since Herodotos, whether merchant or plain citizen, must have been more skilled. My similar mistakes only suggested how Herodotos could have arrived at his result, whatever the reason. Only if the use of the abacus can be proved in the other two cases do we have a presumption that an abacus was also the scene of this mistake.

VII, 187: the problem is how many *medimnoi,* of 48 *choinikes* each, will 5,283,220 men, each consuming a *choinix* a day, eat in one day. Herodotos' solution is 110,340 *medimnoi.* The correct solution may be obtained as follows:

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     110067 plus
  48|5283220
     48
     --
      48
      48
      --
       322
       288
       ---
        340
        336
        ---
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As Schweighaüser [7] pointed out, Herodotos has mistaken the penultimate remainder (340) for the true quotient. Having correctly divided the 528 myriads by 48, he set down the solution: 110,000. This method of breaking down the problem into two stages was seen in the previous calculation (341 generations taken as 300 and 41). The choice of what to add to that first solution from the second calculation is neither difficult nor confusing in our paper-and-pencil system. We must try it on the abacus; but first let us look at the third mistake.

III, 89-95: the problem is more complex. Of the twenty Persian satrapies, nineteen paid the greater part of their tribute in silver, for which the standard was the Babylonian talent; the twentieth satrapy paid in gold dust, for which the standard

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[7] *Ad loc.* Cf. R. W. Macan, *Commentary on Herodotus VII-IX* (London, 1908), *ad loc.* and How and Wells, *op. cit.,* II, pp. 213-214.

was the Euboic talent. The Babylonian talent has 70 [8] Euboic minas.[9] The tributes of the nineteen satrapies are added together and converted from Babylonian talents to Euboic (9540 talents). Unfortunately, Herodotos does not record the total of Babylonian talents before conversion to the Euboic system, so there is no immediate check on the ratio of 60:70. Nor is there any guarantee either that his arithmetic is correct as far as the addition goes or that he added in all the items that he lists.[10] But wherever we can check him fairly (see note 3 above), Herodotos' addition seems to be unimpeachable, and only if it is impossible to trace the steps which he took in his calculations can we assume that he omitted one or more items. The nineteen items add up to a total of 7740 Babylonian talents. Since each Babylonian talent equals 70 Euboic minas and each Euboic talent equals 60 Euboic minas, it is necessary to multiply 7740 by 70 and divide the product by 60 in order to get the Euboic talents from the Babylonian. 70 x 7740 is 541,800. We know from the two previous problems that it was Herodotos' practice to break down his problem into stages. So we shall set it up as follows, taking only the myriads first:

$$\begin{array}{r} 9000 \\ \underline{60}\overline{)540000} \end{array} \qquad\qquad \begin{array}{r} 30 \\ \underline{60}\overline{)1800} \end{array}$$

In adding what should have been the two quotients, Herodotos has again taken the quotient from the myriad division and added to it not the other quotient nor any remainder but the dividend from the first stage of his operations, *i. e.* 540.[11] How this

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[8] Although Reizke's (Th. Mommsen, *Röm. Münzwesen* (Berlin, 1860), pp. 23-24) conjecture to add ὀκτὼ καί (making the Babylonian talent equal to 78 Euboic minas) is usually accepted (cf. How and Wells, *op. cit.*, I, pp. 281-282) there is no good textual reason for any insertion at all. And as far as actual knowledge of the Babylonian weight system is concerned, there is no certainty that would exclude this ratio of 60:70 (cf. O. Viedebantt, "Forschungen zur Metrologie des Altertums," *Sächsische Akademie der Wissenschaften*, Abh. Phil.-Histor. Kl., XXXIV, pp. 114-115). Most efforts to define the various weight systems are complicated by often desperate attempts to derive all the ancient Mediterranean systems from one particular one by various and extremely refined adjustments. Cf. C. F. Lehmann-Haupt, P.W., *R.E.*, Suppl. III, cols. 588-592. For a bibliography of interpretations on this passage of Herodotos and for a typically involved explanation, see Viedebantt, *op. cit.*, pp. 114-120. There seems to be no need here to rehearse the various interpretations, since it is our aim to show that, traced to its source, the difficulty can be dismissed as an error in arithmetical operations.

[9] Herodotos does not state the elementary fact that any talent has 60 minas in its own system, so that the Babylonian talent has 60 Babylonian minas and the Euboic talent has 60 Euboic minas.

[10] It is the contention of those who change the ratio to 60:78 (and of at least one who does not, i. e. Viedebantt, *loc cit.*) that the 140 talents of the fourth satrapy which were used to support the cavalry (III, 90, 3) were not included in the total of Babylonian talents which came to Dareios (Mommsen, *loc. cit.*; Lehmann-Haupt, *loc. cit.*) With a total of 7600 Babylonian talents at the ratio of 60:78 they obtain a total of 9880 Euboic talents and point with triumph to the one manuscript (S^1) which shows 9880 *in rasura*. But since this one reading can have been reached by a copyist subtracting the gold talents given by Herodotos (4680) from his total (14560), it is no evidence of MS tradition (Viedebantt, *loc. cit.*).

[11] This does not explain what appears to be a mistake in addition. Adding 9540 and 4680

was a possible and even an easy thing to do must be explained with reference to calculation with an abacus.

Ancient references to the use of the abacus (see note 1 above) combine with extant examples [12] to illustrate its form and use. It must have columns vertical to the

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(which should give 14220), Herodotos writes: "when all these are added together the total in Euboic talents collected for annual tribute to Dareios was 14560. Letting go what is less than these (talents), I do not count it." Since nothing less than a talent is involved in the two factors mentioned, it is my conviction that what we have here is not a mistake in arithmetic but the inclusion of various tributes-in-kind converted into their cash value: 360 white horses (90, 3); fish and grain from Egypt (91, 2-3); 500 eunuchs (92, 1). For the true total Herodotos must have included these items. Presumably he, or his source, reckoned each in talents or parts of a talent and added them to get something more than 340 talents (the difference between 14220 and 14560). The excess was then dropped to keep the round number.

[12] All of the following have been called abaci, not always with complete justification, and certainly without regard for the possibility that there may have been different types of abaci. Until the functions of the Greek abacus are more clearly understood, they must be studied as a group, Such scanty publication as most of them have enjoyed leaves much to be desired in the way of details. Mr. E. Vanderpool and Mr. M. Mitsos have kindly examined stones in Athens and Oropos in order to answer my questions.

1) *I.G.*, II^2, 2777. Pentelic marble. Length, 1.49 m.; width, 0.754 m.; thickness, 0.045-0.075 m. Height of letters, 0.012-0.016 m. A set of 11 lines are bisected by a middle line; at the intersection of the middle line and third, sixth, and ninth lines there is an X; at the right end of the stone is a set of five lines; on the top surface along three edges of the slab are rows of numerals, one row at each edge; one row readsT𐅆X𐅅H𐅄ΔΓͰICTX; the two others omit the first two signs. See the drawing in *I.G.* or in Daremberg-Saglio, *s.v.* Abacus.

2) *I.G.*, II^2, 2778. Pentelic marble. Length, 0.76 m., width, 0.75 m.; thickness, 0.085 m. Height of letters, 0.004 m. Mutilated at left. No mention of lines. A row of numbers: [M]X𐅅H𐅄ΔΓͰOC..

3) *I.G.*, II^2 2781. Pentelic marble. Length, 1.19 m.; width, 0.49 m.; thickness, 0.075 m. Height of letters, 0.022-0.013 m. On the surface, along one edge, are the numbers:X𐅅H𐅄ΔΓͰTIC. Below the numbers are six circles of various sizes.

4) 'Αρχ. 'Εφ., 1925-1926, pp. 44-45, no. 156. White marble. Length, 1.28 m.; width, 0.78 m.; thickness, 0.085 m. Height of letters, 0.025-0.029 m. Five lines perpendicular to one short side; 11 lines perpendicular to long sides; small semi-circles at either end of the 11-line group and also of the five-line group, attached in each case to the end line at its middle section; X's at the center of the third, sixth, and ninth lines in the 11-line group. A row of numbers along one short side: X𐅅H𐅄ΔΓͰICTX.

5) *Ibid.*, no. 157. White marble slab with rim; right end lost. Length, 0.80 m.; width, 0.64 m.; thickness (including rim), 0.12 m. Height of letters, 0.022 m. Four lines preserved perpendicular to long sides; a row of numbers: MT𐅆X𐅅H𐅄ΔΓ . . .

6) *Ibid.*, no. 158. Uninscribed white marble slab with rim. Length, 1.305 m.; width, 0.645 m.; thickness, 0.163 m. On center of slab, 11 lines, and perhaps 11 lines in opposite corners.

7) *Ibid.*, no. 159. Uninscribed white marble fragment. Two lines preserved.

8) *Ibid.*, no. 160. Uninscribed white marble fragment. Group of five lines preserved; also two other lines.

9) *I.G.,* XII, 7, 282 (Minoa). Two fragments of marble, broken on all sides; columns marked off by lines; at top of each column, a number, e. g. X𐅅H etc.

user (Herodotos, II, 36), unmarked pebbles which may be moved from column to column (Diogenes Laertios, I, 59; Polybios, V, 26); and it is thought of as stone (Aristophanes, *Vespae*, 332). The extant examples of what may be abaci are perhaps naturally of stone, except for the casually converted roof tiles,[13] but otherwise vary considerably in their form. The only thing which all thirteen examples have in common is a flat surface. Ten have a row or rows of numbers, but three,[14] which are otherwise almost identical with some of the ten, have no trace of numbers. Seven have a group or groups of lines which form columns, but six [15] are either not sufficiently preserved to show lines or definitely do not have them. There is always the possibility that either numbers or lines were painted in and are no longer visible, but whether or not that is the case there is certainly the possibility that we are here dealing with at least two different kinds of abacus. This latter possibility may be fruitfully pursued through a consideration of the relative position of lines and numbers on those abaci on which both appear. On only one of these (No. 9 in note 12) do the numbers stand at the head of columns and thus label them. In the others the columns are unlabelled and in this respect similar to the modern oriental abacus.[16] A simple problem in addition will show the operation of the two types:

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10) *I.G.*, XII, 5, 99 (Naxos). Secoma with a row of numbers: X𐅅H𐅄ΔΓ⊢TIC.

11) *I.G.*, IX, 1, 488 (Akarnania). Numbers: M𐅆X𐅅H𐅄 . . .

12) *B.S.A.*, XXVIII, 1926-1927, pp. 144-145. Two edge fragments with center missing (where lines might have been). A row of numbers along two opposite sides: M𐅆X𐅅H𐅄Δ𐅇MΔΓΣ ▷ TOCX.

13) Furtwängler u. Reichhold, *Gr. Vasenmalerei*, II, p. 142. The so-called Darius vase. Table with letters: MΨH ▷ΓO< T. Pebbles in columns below. Accountant with tablet seems to be recording sums done in columns on table. Cf. Heath, *op. cit.*, p. 49; Tod, *B.S.A.*, XVIII, 1911-1912, p. 124.

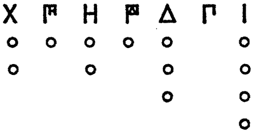
[13] These roof tiles (*I.G.*, II², 2779-2780) with their rows of scratched numbers can only be informal imitations of the "regular models," and do not shed further light on the more complete stone abaci.

[14] Nos. 158-160 from the Amphiareion (Nos. 6-8 in note 12) have no numbers inscribed; although two of these are only fragments, the other appears to be complete.

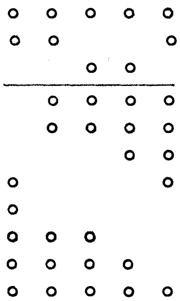
[15] *I.G.*, II², 2781 (No. 3 in note 12) has only circles and no columns. No. 10 is a measure table; No. 12 is not preserved where the lines usually are; Nos. 2, 11, and 13, do not have lines.

[16] Concerning the date of the oriental abacus, L. Carrington Goodrich, who kindly provided me with bibliography on the use of the Chinese and Japanese abaci writes in a letter, "A Chinese scholar, Li Yen, writing in *Yenching Jo.* 10, 2123-38 (in Chinese) a few years ago, reports that the rules for reckoning with the abacus appear in Chinese books dated respectively 1274, 1299, and 1355. So it seems to have penetrated China during the Mongol period." For the working of these abaci, see Dickson H. Leavens, "The Chinese suan-p'an," *Amer. Math. Mo.*, 27, April, 1920; C. G. Knott, "The Abacus in its Historic and Scientific Aspects," reprinted and abridged from *Trans. of the Asiatic Soc. of Japan*, 1886, in E. M. Horsburgh, *Modern Instruments and Methods of Calculation* (London, 1915).

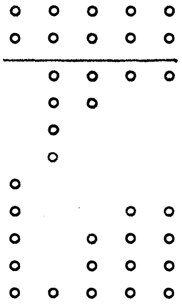
No. 9



Modern Oriental



(For ease of representation vertical lines are omitted: those separating the columns in the ancient example; and the wires on which the beads are strung in the modern abacus.) Both abaci are set up with the number 2784 (XX𐅆HH𐅄ΔΔΔIIII).[17] The number 1427 (XHHHHΔΔΓII) is to be added. No. 9, like all those from ancient Greece, has removable pebbles, so that only those actually required for the particular calculation appears in the columns. The modern abacus has five beads on each of the wires in the lower section, each bead representing one unit in that decimal position, and two in the upper section, each representing five of the units below;[18] those in use are pushed to the middle bar. Let us add 1427, taking the modern abacus first and remembering to start at the left (Herodotos, II, 36; cf. bibliography for modern methods in note 16). 1(000) is added to 2(000); 4(00), added to 7(00), gives 11(00) and so requires the addition of another unit in the thousand column; 2(0) and 8(0) mean a similar shift, as do also 4 and 7. The completed sum appears as follows:



Since on each wire there are the equivalent of 15 units, often the addend may simply be added on in beads, and the then visible total may be resolved. But it is more efficient to do part of the operation mentally, saying 4 and 7 are 11, so that one bead is added in the column to the left and only one retained in this particular column.

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[17] When money is in question the unit sign for a drachma (⊢) is used; for non-monetary representation, the unit is usually a simple stroke. Cf. M. N. Tod, *B.S.A.*, XVIII, 1911-1912, p. 132.

[18] This is the Chinese type; the Japanese abacus has only four unit-beads and one "five"-bead.

With the ancient abacus No. 9 we take the addend in Greek numbers (XHHHH ΔΔ𐅃II) and add a pebble for each of these symbols in the appropriate columns:

Addition

X	𐅅	H	𐅄	Δ	𐅃	I
o	o	o	o	o		o
o		o		o		o
				o		o
						o
o		o		o	o	o
		o		o		o
		o				
		o				

Resolution

X	𐅅	H	𐅄	Δ	𐅃	I
o		o		o		o
o		o				
o						
o						

The resolution moves from right to left thus: the six pebbles under I become one under 𐅃 and one under I; the two pebbles under 𐅃 become one under Δ; the six under Δ become one under 𐅄 and one under Δ; the two under 𐅄 become one under H; the seven under H become one under 𐅅 and two under H; the two under 𐅅 become one under X, thus making a total of XXXXHHΔI. On this abacus each pebble stands for a symbol and no arithmetical adding need be done; one merely adds pebbles and then goes through removing sets of pebbles which are numerous enough to appear in the next larger column. The same principle is involved as in the columns of 10 divided into five units and two "fives," but it works in a more primitive fashion and requires no mental arithmetic such as we know the Greeks used (Aristophanes, *Vespae*, 656). But where the columns are not labelled, as is the case with the majority of extant abaci, the alternation of decimal and quinary columns would be confusing for this type of calculation. (The abaci with numbers but no lines seem to me to belong to the same category as No. 9; on these, pebbles may be arranged under the various symbols; pebbles may be added or taken away; and totals of the lesser units may be transferred to the larger units. It is this which the clerk on the Darius vase appears to be doing.)

Of the abaci with unlabelled columns, the most elaborate (No. 1 in note 12) has a middle line bisecting the group of 11 lines. This structural similarity with the modern abacus suggests that there was functional similarity as well, and that the columns were purely decimal with fives above and units below. The most recent discussions of the Greek abacus are by Nagl,[19] who changed his views [20] concerning the operation of the abacus after the discovery of No. 9. Previously he had assumed [21] that the middle line of the Salamis abacus (No. 1) divided the fives from the units in purely decimal columns. Then, forced, as he declares, by the abacus from Minoa (No. 9) to assume that this was *the* type of Greek abacus, he worked out methods of

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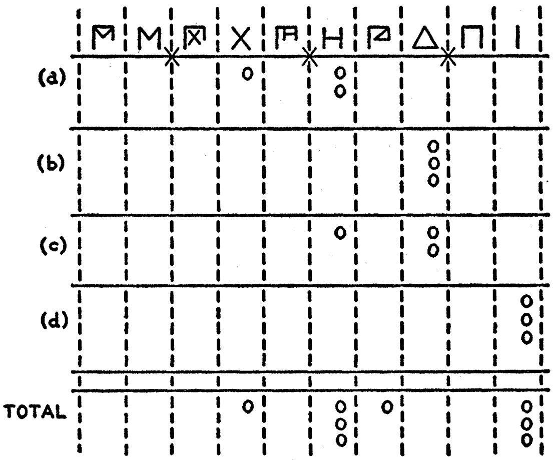
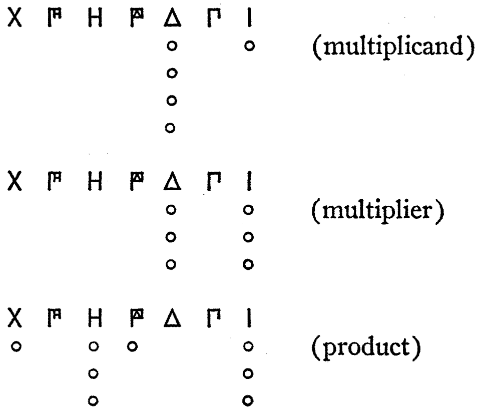
[19] *Sitzb. Ak. Wien.*, 177, 5; P.W., *R.E.*, Suppl. III, *s.v.* Abacus.

[20] *Sitzb. Ak. Wien.*, 177, 5, p. 28.

[21] *Abhandl. zur Geschichte der Math.*, IX, 1899, in *Zeitschrift für Math. u. Physik.*

calculation for the alternating decimal and quinary columns. In his example of multiplication[22] he shows the complexities of that system, which we can paraphrase here by doing Herodotos' calculation of 33 x 41 (II, 142). Nagl uses the form of the Salamis abacus but assumes that the columns were thought of as labelled like those on No. 9. We shall label the columns for the sake of clearness and follow Nagl in depicting the various stages by a sort of " running abacus "; for ease of representation the three rows of numerals will not be placed along the three edges as in the abacus but simply listed at the side.





a) 30 x 40. Here Nagl would invoke the " rule by position " of Archimedes:[23] " The position of the product of two numbers is equal to the sum of their positions less one." That is, the number 12 (positions 00 or ΔΙ) times 970 (positions 000 or HΔΙ) gives a product position of 2 plus 3 minus 1 equals 4 (0000 or XHΔΙ). The product of 1 and 9 (the numbers in the first position of each factor) will be 9 and will be located in the 4th position (thousands or X). If the multiplier is 82 instead of 12, the product positions will still be 4 (X), and the product of 8 x 9 (the first numbers) will be found based on position 4 (X), but starting in position 5, i. e., 72(000) or 𐅇MMXX. In the combination quinary and decimal system this rule is modified[24] so that 33 (ΔΔΔΙΙΙ) and 41 (ΔΔΔΔΙ) each have 3 positions (ΔΓΙ); 3 plus 3 minus 1 equals 5 (H𐅄ΔΓΙ), so that the 2 of the 12 (3 x 4) will fall in the 5th column (H) on an abacus like No. 9; the 1 will fall in the 7th position (X). So here 3(0) x 4(0) (ΔΔΔ x ΔΔΔΔ) equals 1200 (XHH) and we shall place 1 pebble in the X column and 2 pebbles in the H column.

b) 30 x 1. The position rule is 3 plus 1 minus 1 equals 3 (ΔΓΙ). Three pebbles

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[22] *Sitzb. Ak. Wien.*, 177, 5, pp. 56 ff.

[23] *Ibid.*, pp. 49 ff.; Archimedes, ψαμμίτης III, ed. Heiberg, II, 240.

[24] *Sitzb. Ak. Wien.*, 177, 5, p. 54.

will go into the Δ column. The multiplicand has been completely multiplied by the first element of the multiplier.

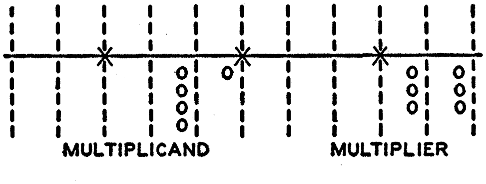
c) 3 x 40. The position rule is 1 plus 3 minus 1 equals 3. Two pebbles will go into the Δ column, and one into the H column.

d) 3 x 1. The position rule is 1 plus 1 minus 1 equals 1; three pebbles will go into the I column.

Total: the one pebble in the X column remains; the two and one in the H column are added; the three and two in the Δ column are replaced by one in the 𐅄 column; and the three pebbles in the I column remain. The product is XHHH𐅄III (1353).

Now let us try the same multiplication on the same abacus (No. 1), but treat it as if it were a modern abacus with the middle line dividing purely decimal columns into fives and units. Here we place both multiplicand and multiplier in the columns, making use of the X's at the intersections of the third, sixth, and ninth lines with the middle line.[25] They can serve as helpful indicators like the movable pointer on my Chinese-type abacus. The multiplier will go in the columns at the extreme right; and the multiplicand may be placed so that the middle X marks its end. Where a number ends in one or more zeros and the columns are not labelled, some device of this sort is necessary to assure that one gets the full value of the number.[26] The product, by this system, will replace the multiplicand, and thus the middle X will mark its last position. Position, by this method, is determined by much the same rule as that used by Nagl, but here it is applied directly to the columns, or is, more accurately, a function of the columns. A two-position multiplier requires that the product be started in the first or second column to the left of the multiplicand, depending on whether the product of the first members in each factor is composed of one or two positions. In the case of a multiplicand 321 and a multiplier of 32, the product of the first two numbers is 9 and so goes in the first column to the left; with 30 x 40 (as here) the product of the first digits is 12 and so begins in the second column to the left. Where the multiplier has three positions, the product will start in the second or third column to the left of the multiplicand, etc.

The problem is set up thus:



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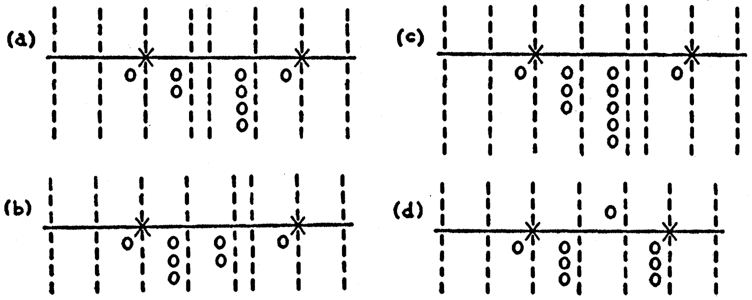
[25] These X's occur on both No. 1 and No. 4 in note 12.

[26] This use of the X's will be examined further in connection with problems in division. See below, p. 283.

For the sake of brevity and clarity the following figures will give only the section of the abacus in which the multiplicand, and the product into which the multiplicand is being converted, appear. In order to keep clear the distinction between product and as yet unmultiplied multiplicand, I put a double line between. This would have been marked with another pebble by the Greeks.



a) 30 x 40 equals 1200.

b) 3 x 40 equals 120. The first digit of the multiplicand, having been completely multiplied, can be replaced by the product, in accordance with the position-rule that a multiplier of one position requires that a two-position product begin one column to the left and replace the multiplicand with its second digit and that a one-position product simply replace the multiplicand.

c) 30 x 1. The position-rule for a two-position multiplier where the product of the first digits occupies only one position places the three pebbles in the first column to the left of the multiplicand.

d) 3 x 1. The multiplicand, completely multiplied, is replaced by the product. In the " tens " column the five units are resolved to one " five " in the upper section. The product is complete: 1353.

Again, as with the example of addition, it is seen that the purely decimal columns with the middle line allow a technique that makes of the abacus a true calculating machine, whereas the quinary-decimal labelled columns can be used only as a scoreboard to record results or as a simple counting board. Unfortunately, Nagl has not given us an example of long division, but his suggestions as to how it should be handled show that for this he is obliged to use the columns as something more than a scoreboard. The dividend, he says,[27] must be placed in the columns and, while the multiplication of quotient by divisor is carried on in the head or on the fingers, the product is subtracted from the dividend in the columns. This supports our previous conclusion that labelled columns allow of no other calculations besides addition and subtraction and can only serve to record results. Nagl's system, moreover, requires that each

[27] *Sitzb. Ak. Wien.*, 177, 5, p. 65.

quotient, as it is arrived at, be placed under whichever row of numerals has been left open for it; that quotient can not be collected in the columns and moved as a whole. It is on this point that we can once again establish contact with Herodotos, whose mistakes in long division require a method of calculation in which the quotient as a whole [28] is taken off columns in which the remainder is also shown, so that confusion between the two is possible. Being forced by Herodotos' mistakes to use the purely decimal, unlabelled columns of the Salamis abacus (No. 1) for Herodotos' calculations, we must conclude that there were two kinds of abacus in use in ancient Greece. It is neither right nor necessary to assume that the Salamis abacus with its unlabelled and transected columns would be operated in the same way as the Minoa counting-board (No. 9) with its labelled columns. The latter is a scoreboard which is completely adequate only for straightforward addition and subtraction of pebbles, but the former is a stream-lined machine for more complicated operations.[29] It is wrong, again, to assume that because the Greeks recorded numbers with a combination of the quinary and decimal systems that their abaci must have alternating quinary and decimal columns. What was convenient in the recording of numbers (to avoid as many as nine units in a row) was not necessarily convenient for the abacus.

We shall therefore work out Herodotos' problems and mistakes in long division on the unlabelled transected columns of the Salamis abacus. And in using it we must note that it was apparently made for monetary calculations, since all of the rows of numbers include the fractions of the drachma, and the one row which goes higher than 1000 (X) has as its first figure the T (talent). There is nothing to have prevented Herodotos from using such a one, but it is perhaps more likely that he used something with M (myriad) substituted for T (such as Nos. 5 and 11 in note 12), without fractions added after the unit and with a simple unit sign (I) rather than the drachma sign (⊢). It may be mentioned in passing that for the addition of the contingents of the Persian army (VII, 184 ff.) the ten columns of the Salamis abacus would have been adequate only if they are interpreted as purely decimal, since on ten alternating quinary-decimal columns the top number possible is only 100,000. And ten columns seem to have been standard, if we are to judge from the three examples on which whole sets of lines are preserved (Nos. 1, 4, 6 in note 12).

First, the problem of 5,283,220 Persian army personnel eating 5,283,220 *choinikes* of grain a day. How many *medimnoi* does this represent, at 48 *choinikes* to the *medimnos*? As Herodotos writes it (VII, 187) the dividend is 528 myriads,

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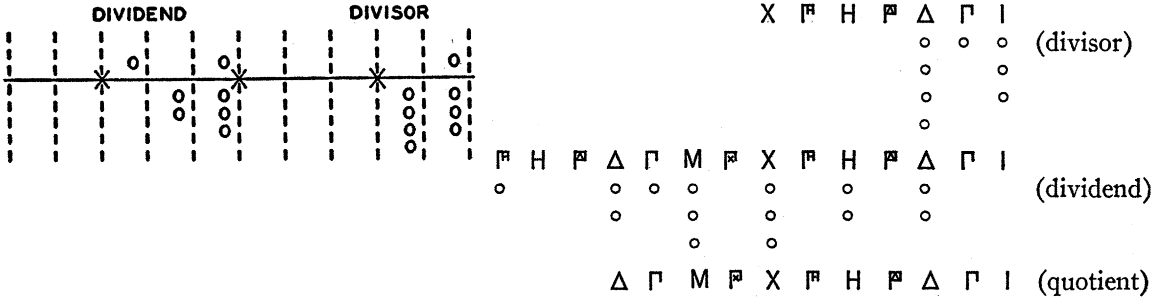
[28] That is, the whole quotient of each of the two operations: the myriad division, and the division of what is less than a myriad.

[29] It may be noted here that Nagl's explanation of the middle line (*op. cit.*, p. 55) is not particularly convincing: that the upper section may be used for a second calculation, and that the upper section may hold the product of an integral calculation while the fractional calculation is being carried on below.

3 thousands, 2 hundreds, and 2 decades. And as we saw good reason to think (p. 273), he broke the whole number down at least into 528 myriads and 3,220 before dividing. This is especially suitable to the Greek number system and the abacus, where no number greater than the myriad has a number or symbol, so that this number written in figures would be 𐅅ΔΔΓMMMXXXHHΔΔ.[30] The myriads, like talents in monetary numbers, are treated as units. On the abacus they must also have been treated as units and divided in a separate operation.

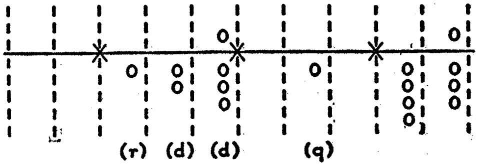
With the abacus set up for this problem, it should look as follows:



The whole dividend and divisor are recorded under the rows of numerals; in the columns the first dividend (528) is placed so that the middle X marks its end.[31] The divisor is also placed in the columns, since this juxtaposition of the two factors seems to increase the value of the abacus as an efficiently operating machine.

The steps of the operation follow: [32]

a) 4 goes into 5 with a remainder of 1; 4 pebbles are removed and 1 remains; the quotient of 1 is put in another column out of the way. There seems at first no reason why it should not immediately be taken out of the columns and put under the row of numbers where the quotient will come out. But where, as here, there is room to keep the quotient on tap, it is better to do so, since the first quotient tried sometimes has to be reduced and so is not always final. As for the exact location of it



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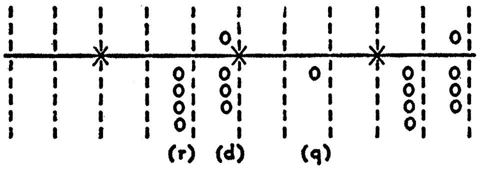
[30] Cf. M. N. Tod, *B.S.A.*, XVIII, 1911-1912, p. 128, citing *I.G.*, XI, 2, 203, B 101.

[31] If myriads were always treated as units and dealt with separately, no dividend would ever have more than four positions, and so could always use one of the X's as its base line and still leave room for the divisor, if not always for the quotient.

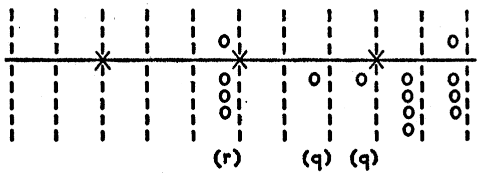
[32] As aids to understanding the figures, the following letters will be placed under the columns: r for remainder; d for dividend; q for quotient.

in the columns, by the rule of position in division,[33] the number of positions in the quotient is equal to the number of positions in the dividend minus the number of positions in the divisor plus 1. So here: 3 minus 2 plus 1 equals 2. The number of myriads in the quotient of 528 divided by 48 will have two positions. Here the X to the right is made use of as an end-limit for the quotient and the first figure of the quotient is put in the second column to the left of that X.

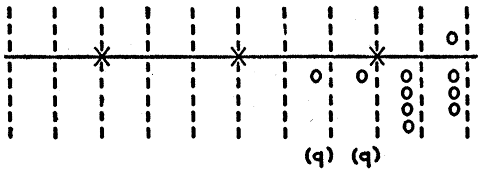
b) The second figure of the divisor times the quotient: 8 x 1 equals 8. Subtract 8 from 12 (which is part remainder and part dividend), leaving a remainder of 4:



c) The first remaining figure of the remainder-dividend combination divided by the first figure of the divisor: 4 divided by 4 equals 1, with no remainder. The four pebbles are removed, and one pebble, as the second quotient, goes into the second quotient column:



d) The second figure of the divisor times the second quotient: 8 x 1 equals 8; eight pebbles are removed and there is no remainder.



The division of 528 myriads by 48 is complete. To clear the columns for the next stage of the division, the quotient (11 myriads) is taken out and put under the row of figures saved for the quotient: $\underset{\circ}{\Delta}$ Γ $\underset{\circ}{\mathrm{M}}$ 𐅆 X 𐅅 H 𐅄 Δ Γ I

Now we arrange the board for the division of 3220 by 48. Since 3220 ends with a zero (or empty position), the middle X serves to mark its true end. The position

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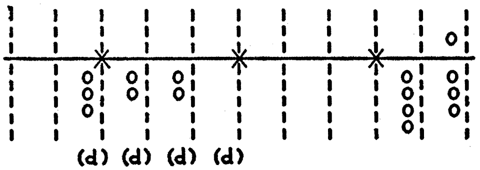
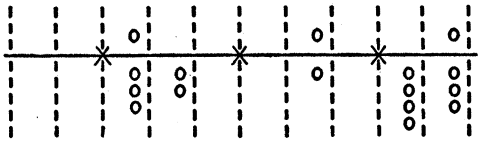
[33] Opposite to that for multiplication. See Nagl, *op. cit.*, p. 64. Of course, where the first number of the divisor will not go into the first number of the dividend, the quotient will have its first position empty.

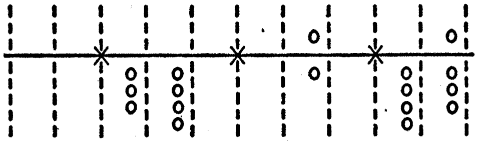
rule (4 minus 2 plus 1 equals 3) requires that the quotient have three positions. The three columns marked off by the middle and righthand X may be used.



a) Since 4 will not go into 3, the first position of the quotient will be empty. 4 will go into 30 seven times, but to avoid a false quotient we remember that 48 is nearer 50 than 40, so we try a quotient of 6; 6 x 4 equals 24; pebbles representing the 24 may then be subtracted from the 30, leaving a remainder of 6, which will be added to the 2 of the dividend in the next column. (It may be easier to think of the 24 being subtracted from the 32, leaving a remainder of 8.) The quotient 6 is put in the second quotient column.

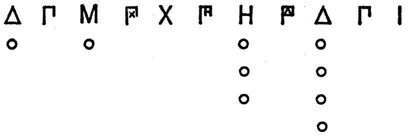


b) Second figure of the divisor times quotient: 8 x 6 equals 48; pebbles representing 48 may be subtracted from 82, leaving a remainder of 34; this subtraction requires that one pebble from the column which has 8 be broken down into 10 for the column to its right, so that these two columns have 7 and 12. Then 4 from 7 leaves 3; and 8 from 12 leaves 4.



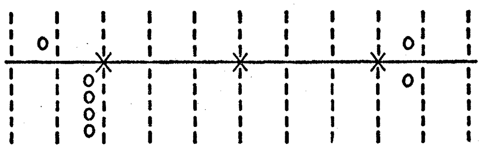
It is at this point that Herodotos stopped, perhaps because he forgot and assumed that 060 rather than 340 was the remainder and so not worth worrying about.[34] It was still necessary to take the quotient from the columns to the row of figures saved for the quotient. According to the position rule, the quotient should have three positions. Herodotos, forgetting that the first position of the quotient was empty, and laboring under the illusion that 060 was the remainder, simply took the three-position figure nearest to hand, *i. e.* 340, which he added to the quotient row of numerals:

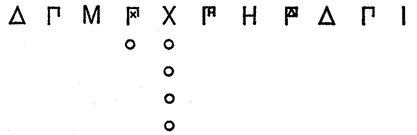
[34] With the X's used to mark end limits of dividend and quotient there was no set place for either. A four-position dividend might use the right-hand X as its base line if the quotient were also to have four positions. With no set place for the quotient, confusion would be quite easy.



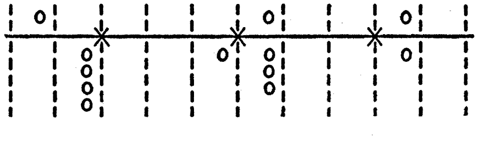
I think it is right to say that it was the emptiness of the first position in the quotient which confused Herodotos. The same situation occurs in III, 95, where he again avoids the true quotient with its empty first position in the second stage of division after the myriads have been disposed of. That is, Herodotos' quotient (9540) consists of 9(000) from the myriad division and 540 as the quotient from the second part of the division. But the correct second quotient is 030. The problem was to divide 541,800 (i. e., 70 x 7740) by 60. The steps are as follows:

a) 54 myriads divided by 60. Position rule: 2 minus 2 plus 1 equals 1. And if 6 went into 5, the quotient would be in myriads. But since it does not, the quotient will not be in myriads, but in the next lower position, i. e., thousands. Nine is the correct quotient without any remainder, and so is immediately put in the quotient row of numerals:

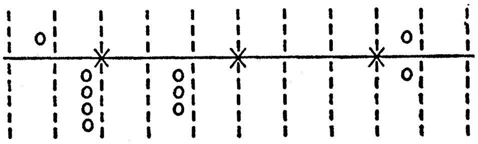




b) Then, instead of clearing the board of the myriad dividend, Herodotos must have left the 54 and put down the 1800 wtih its end against the right-hand X, planning to put the quotient in the columns between the left-hand X and that in the middle, from which the dividend would almost immediately be divided up and removed. 6 will not go into 1, so the first position of a three-position quotient (4 minus 2 plus 1) will be left vacant.



c) 6 will go into 18, leaving no remainder, and the quotient of 3 will go into the second quotient column. Herodotos, then wishing to take out the quotient, ignored the 030 as a negligible remainder and took 540 (i. e., 54 rounded out to make a three-position quotient) to make the complete quotient: 9540.



The mistake in Herodotos, II, 142, of subtracting instead of adding need not be abacus-inspired, but since it also involves some confusion about the relation of the various factors with one another it too may be thought of as taking place on an abacus. There can, of course, be no guarantee that the particular motives and reasons assigned to Herodotos here were those responsible for his mistakes, but it does seem clear that an abacus of the Salamis type provides the most favorable set of circumstances for those mistakes.

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CORRIGENDA

p. 4, line 38: read Ὄαθεν (?).

p. 5, line 41: read Ὄ̣α̣[θεν (?).

p. 18, line 8 of S7: read ὠνη.

p. 21, in the Index: read Ἁγνίας Ὄ̣α̣[θεν (?)]; read [Διό]φαντος.

p. 27, line 47: read ἑαυτῶν.

p. 37, at the beginning of the paragraph: for Line 1 read Lines 2, 42.

p. 41: for Lines 59-60, 135-137 read Lines 59-60, 136-138; under item e for (169/8-156/5) read (*ca.* 173/2-161/0); in the last line of text, under item c, for (*ca.* 176/5-169/8) read (*ca.* 176/5-170/69).

p. 42: under item e for (*ca.* 169/8-156/5) read (*ca.* 173/2-161/0); under items j and k for (159/8) read (164/3).

p. 44: for Line 83 read Line 82.

p. 46: at the top of the page, for Herakleitos read Asklepiades.

p. 53, note 5: for typographical read clerical.

p. 59, line 6 of the Greek text: for ἀγ]αθεῖ read ἀγ[αθεῖ.

p. 60, line 44: for ἀπαινέσαι read ἐπαινέσαι.

p. 71, line 2 of the first Greek text: for Σω[κράτου read Σωκρ[άτου.

p. 89, Col. I, third and fourth lines from the bottom: for Ἀπόλ[ωνι – –] read Ἀπόλ[λωνι – –].

p. 95, note on 170/69: for p. 35 read p. 38, note 28.